

SUMS WITH MULTIPLICATIVE FUNCTIONS OVER A BEATTY SEQUENCE*

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Abstract

We study sums with multiplicative functions that take values over a non-homogenous Beatty sequence. We then apply our result in a few special cases to obtain asymptotic formulas such as the number of integers in a Beatty sequence representable as a sum of two squares up to a given magnitude.

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1 Introduction

Let $A \geq 1$ be an arbitrary constant, and let \mathcal{F}_A be the set of multiplicative functions such that $|f(p)| \leq A$ for all primes p , and

$$\sum_{n \leq N} |f(n)|^2 \leq A^2 N \quad (N \in \mathbb{N}). \quad (1)$$

Exponential sums of the form

$$S_{\alpha, f}(N) = \sum_{n \leq N} f(n) e(n\alpha) \quad (\alpha \in \mathbb{R}, f \in \mathcal{F}_A), \quad (2)$$

where $e(z) = e^{2\pi iz}$ for all $z \in \mathbb{R}$, occur frequently in analytic number theory. Montgomery and Vaughan have shown (see [7, Corollary 1]) that the upper bound

$$S_{\alpha, f}(N) \ll_A \frac{N}{\log N} + \frac{N(\log R)^{3/2}}{R^{1/2}} \quad (3)$$

holds uniformly for all $f \in \mathcal{F}_A$ provided that $|\alpha - a/q| \leq q^{-2}$ with some reduced fraction a/q for which $2 \leq R \leq q \leq N/R$. They also proved that this bound is sharp apart from the logarithmic factor in R . In this paper, we use the Montgomery-Vaughan result to estimate sums of the form

$$G_{\alpha, \beta, f}(N) = \sum_{\substack{n \leq N \\ n \in \mathcal{B}_{\alpha, \beta}}} f(n), \quad (4)$$

where $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, $f \in \mathcal{F}_A$, and $\mathcal{B}_{\alpha, \beta}$ is the *non-homogenous Beatty sequence* defined by

$$\mathcal{B}_{\alpha, \beta} = \{n \in \mathbb{N} : n = \lfloor \alpha m + \beta \rfloor \text{ for some } m \in \mathbb{Z}\}.$$

Our results are uniform over the family \mathcal{F}_A and nontrivial whenever

$$\lim_{N \rightarrow \infty} \frac{\log N}{N \log \log N} \left| \sum_{n \leq N} f(n) \right| = \infty,$$

a condition which guarantees that the error term in Theorem 1 is smaller than the main term. One can remove this condition, at the expense of losing uniformity with respect to f , and still obtain Theorem 1 for any

bounded arithmetic function f (not necessarily multiplicative) for which the exponential sums in (2) satisfy

$$S_{\alpha,f}(N) = o\left(\sum_{n \leq N} f(n)\right) \quad (N \rightarrow \infty).$$

The general problem of characterizing functions for which this relation holds appears to be rather difficult; see [1] for Bachman's conjecture and his related work on this problem.

We shall also assume that α is irrational and of finite type τ . For an irrational number γ , the type of γ is defined by

$$\tau = \sup\{t \in \mathbb{R} : \liminf_{n \rightarrow \infty} n^t \llbracket \gamma n \rrbracket = 0\},$$

where $\llbracket \cdot \rrbracket$ denotes the distance to the nearest integer. *Dirichlet's approximation theorem* implies $\tau \geq 1$ for every irrational number γ . According to theorems of Khinchin [5] and of Roth [9, 10], $\tau = 1$ for *almost all* real numbers (in the sense of the Lebesgue measure) and *all* irrational algebraic numbers γ , respectively; also see [2, 12].

Our main result is the following:

Theorem 1. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and suppose that α is irrational and of finite type. Then, for all $f \in \mathcal{F}_A$ we have*

$$G_{\alpha,\beta,f}(N) = \alpha^{-1} \sum_{n \leq N} f(n) + O\left(\frac{N \log \log N}{\log N}\right),$$

where the implied constant depends only on α and A .

The following corollaries are immediate applications of Theorem 1:

Corollary 1. *The number of integers not exceeding N that lie in the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ and can be represented as a sum of two squares is*

$$\#\{n \leq N : n \in \mathcal{B}_{\alpha,\beta}, n = \square + \square\} = \frac{CN}{\alpha \sqrt{\log N}} + O\left(\frac{N \log \log N}{\log N}\right)$$

where

$$C = 2^{-1/2} \prod_{p \equiv 3 \pmod{4}} (1 - p^{-2})^{-1/2}. \quad (5)$$

To state the next result, we recall that an integer n is said to be k -free if $p^k \nmid n$ for every prime p .

Corollary 2. *For every $k \geq 2$, the number of k -free integers not exceeding N that lie in the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ is*

$$\#\{n \leq N : n \in \mathcal{B}_{\alpha,\beta}, n \text{ is } k\text{-free}\} = \alpha^{-1} \zeta^{-1}(k) N + O\left(\frac{N \log \log N}{\log N}\right)$$

where $\zeta(s)$ is the Riemann zeta function.

Finally, we consider the average value of the number of representations of an integer from a Beatty sequence as a sum of four squares. Our result is the following:

Corollary 3. *Let $r_4(n)$ denote the number of representations of n as a sum of four squares. Then,*

$$\sum_{\substack{n \leq N \\ n \in \mathcal{B}_{\alpha,\beta}}} r_4(n) = \frac{\pi^2 N^2}{2\alpha} + O\left(\frac{N^2 \log \log N}{\log N}\right),$$

where the implied constant depends only on α .

Any implied constants in the symbols O and \ll may depend on the parameters α and A but are absolute otherwise. We recall that the notation $X \ll Y$ is equivalent to $X = O(Y)$.

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2 Preliminaries

2.1 Discrepancy of fractional parts

We define the *discrepancy* $D(M)$ of a sequence of real numbers $b_1, b_2, \dots, b_M \in [0, 1)$ by

$$D(M) = \sup_{\mathcal{I} \subseteq [0,1)} \left| \frac{V(\mathcal{I}, M)}{M} - |\mathcal{I}| \right|, \quad (6)$$

where the supremum is taken over all possible subintervals $\mathcal{I} = (a, c)$ of the interval $[0, 1)$, $V(\mathcal{I}, M)$ is the number of positive integers $m \leq M$ such that $b_m \in \mathcal{I}$, and $|\mathcal{I}| = c - a$ is the length of \mathcal{I} .

If an irrational number γ is of finite type, we let $D_{\gamma, \delta}(M)$ denote the discrepancy of the sequence of fractional parts $(\{\gamma m + \delta\})_{m=1}^M$. By [6, Theorem 3.2, Chapter 2] we have:

Lemma 1. *For a fixed irrational number γ of finite type τ and for all $\delta \in \mathbb{R}$ we have:*

$$D_{\gamma, \delta}(M) \leq M^{-1/\tau + o(1)} \quad (M \rightarrow \infty),$$

where the function defined by $o(\cdot)$ depends only on γ .

2.2 Numbers in a Beatty sequence

The following is standard in characterizing the elements of the Beatty sequence $\mathcal{B}_{\alpha, \beta}$:

Lemma 2. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and set $\gamma = \alpha^{-1}$, $\delta = \alpha^{-1}(1 - \beta)$. Then, $n = \lfloor \alpha m + \beta \rfloor$ for some $m \in \mathbb{Z}$ if and only if $0 < \{\gamma n + \delta\} \leq \gamma$.*

From Lemma 2, an integer n lies in $\mathcal{B}_{\alpha, \beta}$ if and only if $n \geq 1$ and $\psi(n) = 1$, where ψ is the periodic function with period one whose values on the interval $(0, 1]$ are given by

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x \leq \gamma; \\ 0 & \text{if } \gamma < x \leq 1. \end{cases}$$

We wish to approximate ψ by a function whose Fourier series representation is well behaved. This will give rise to the fore mentioned exponential sum $S_{\alpha, f}(N)$. To this end we use the result of Vinogradov (see [14, Chapter I, Lemma 12]), which states that for any Δ such that

$$0 < \Delta < \frac{1}{8} \quad \text{and} \quad \Delta \leq \frac{1}{2} \min\{\gamma, 1 - \gamma\},$$

there exists a real-valued function Ψ with the following properties:

- (i) Ψ is periodic with period one;
- (ii) $0 \leq \Psi(x) \leq 1$ for all $x \in \mathbb{R}$;

- (iii) $\Psi(x) = \psi(x)$ if $\Delta \leq \{x\} \leq \gamma - \Delta$ or if $\gamma + \Delta \leq \{x\} \leq 1 - \Delta$;
- (iv) Ψ can be represented by a Fourier series:

$$\Psi(x) = \sum_{k \in \mathbb{Z}} g(k) \mathbf{e}(kx),$$

where $g(0) = \gamma$, and the Fourier coefficients satisfy the uniform bound

$$g(k) \ll \min \{|k|^{-1}, |k|^{-2} \Delta^{-1}\} \quad (k \neq 0). \quad (7)$$

3 Proofs

3.1 Proof of Theorem 1

Using Lemma 2, we rewrite the sum (4) in the form

$$G_{\alpha, \beta, f}(N) = \sum_{n \leq N} f(n) \psi(\gamma n + \delta).$$

Replacing ψ by Ψ we have

$$G_{\alpha, \beta, f}(N) = \sum_{n \leq N} f(n) \Psi(\gamma n + \delta) + O \left(\sum_{n \in \mathcal{V}(\mathcal{I}, N)} f(n) \right), \quad (8)$$

where $\mathcal{V}(\mathcal{I}, N)$ is the set of positive integers $n \leq N$ for which

$$\{\gamma n + \delta\} \in \mathcal{I} = [0, \Delta) \cup (\gamma - \Delta, \gamma + \Delta) \cup (1 - \Delta, 1).$$

Since $|\mathcal{I}| = 4\Delta$, it follows from Lemma 1 and the definition (6) that

$$|\mathcal{V}(\mathcal{I}, N)| \ll \Delta N + N^{1-1/(2\tau)},$$

where we have used the fact that α and γ have the same type τ . Thus, taking (1) into account, we have by the Cauchy inequality:

$$\begin{aligned} \left| \sum_{n \in \mathcal{V}(\mathcal{I}, N)} f(n) \right| &\leq |\mathcal{V}(\mathcal{I}, N)|^{1/2} \left(\sum_{n \leq N} |f(n)|^2 \right)^{1/2} \\ &\ll \Delta^{1/2} N + N^{1-1/(4\tau)}. \end{aligned} \quad (9)$$

Next, let $K \geq \Delta^{-1}$ be a large real number (to be specified later), and let Ψ_K be the trigonometric polynomial given by

$$\Psi_K(x) = \sum_{|k| \leq K} g(k) \mathbf{e}(kx) = \gamma + \sum_{0 < |k| \leq K} g(k) \mathbf{e}(kx) \quad (x \in \mathbb{R}). \quad (10)$$

Using (7) we see that the estimate

$$\Psi_K(x) = \Psi(x) + O(K^{-1}\Delta^{-1})$$

holds uniformly for all $x \in \mathbb{R}$; therefore,

$$\sum_{n \leq N} f(n) \psi(\gamma n + \delta) = \sum_{n \leq N} f(n) \Psi_K(\gamma n + \delta) + O(K^{-1}\Delta^{-1}N), \quad (11)$$

where we have used the bound $\sum_{n \leq N} |f(n)| \ll N$, which follows from (1).

Combining (8), (9), (10) and (11) we derive that

$$G_{\alpha, \beta, f}(N) = \gamma \sum_{n \leq N} f(n) + H(N) + O(K^{-1}\Delta^{-1}N + \Delta^{1/2}N + N^{1-1/(4\tau)}),$$

where

$$H(N) = \sum_{0 < |k| \leq K} g(k) \mathbf{e}(k\delta) S_{k\gamma, f}(N).$$

Put $R = (\log N)^3$. We claim that, if N is sufficiently large, then for every k in the above sum there is a reduced fraction a/q such that $|\alpha - a/q| \leq q^{-2}$ and $R \leq q \leq N/R$. Assuming this for the moment, (3) implies that

$$S_{k\gamma, f}(N) \ll \frac{N}{\log N} \quad (0 < |k| \leq K),$$

and using (7) we deduce that

$$H(N) \ll \frac{N \log K}{\log N}.$$

Therefore,

$$G_{\alpha, \beta, f}(N) - \gamma \sum_{n \leq N} f(n) \ll \frac{N \log K}{\log N} + K^{-1}\Delta^{-1}N + \Delta^{1/2}N.$$

To balance the error terms, we choose

$$\Delta = (\log N)^{-2} \quad \text{and} \quad K = \Delta^{-3/2} = (\log N)^3,$$

obtaining the bound stated in the theorem.

To prove the claim, let k be an integer with $0 < |k| \leq K = (\log N)^3$, and let $r_i = a_i/q_i$ be the i -th convergent in the continued fraction expansion of $k\gamma$. Since γ is of finite type τ , for every $\epsilon > 0$ there is a constant $C = C(\gamma, \epsilon)$ such that

$$C(|k|q_{i-1})^{-(\tau+\epsilon)} < \llbracket \gamma |k|q_{i-1} \rrbracket \leq |\gamma |k|q_{i-1} - a_{i-1}| \leq q_i^{-1}.$$

Put $\epsilon = \tau$, and let j be the least positive integer for which $q_j \geq R$ (note that $j \geq 2$). Then,

$$R \leq q_j \ll (|k|q_{i-1})^{2\tau} \leq (KR)^{2\tau} = (\log N)^{6\tau},$$

and it follows that $R \leq q_j \leq N/R$ if N is sufficiently large, depending only on α . This concludes the proof.

3.2 Proof of Corollary 1

Let $f(n)$ be the characteristic function of the set of integers that can be represented as a sum of two squares. Then Corollary 1 follows immediately from Theorem 1 and the asymptotic formula (see for example [11]):

$$\sum_{n \leq N} f(n) = \frac{CN}{(\log N)^{1/2}} + O\left(\frac{N}{(\log N)^{3/2}}\right),$$

where C is given by (5).

3.3 Proof of Corollary 2

Fix $k \geq 2$ and let $f(n)$ be the characteristic function of the set of k -free integers. Then Corollary 2 follows from Theorem 1 and the following estimate of Gegenbauer [3] for the number of k -free integers not exceeding N :

$$\sum_{n \leq N} f(n) = \zeta^{-1}(k)N + O(N^{1/k}).$$

3.4 Proof of Corollary 3

Put $f(n) = r_4(n)/(8n)$. From Jacobi's formula for $r_4(n)$, namely

$$r_4(n) = 8(2 + (-1)^n) \sum_{\substack{d|n \\ d \text{ odd}}} d \quad (n \geq 1),$$

it follows that $f(n)$ is multiplicative, and $f(p) \leq 3/2$ for every prime p . Moreover, using the formula of Ramanujan [8] (see also [13]):

$$\sum_{n \leq N} \sigma^2(n) = \frac{5}{6} \zeta(3) N^3 + O(N^2 (\log N)^2),$$

we have by partial summation:

$$\sum_{n \leq N} |f(n)|^2 \leq \sum_{n \leq N} \frac{\sigma^2(n)}{n^2} = \frac{5}{2} \zeta(3) N + O((\log N)^3).$$

Therefore, $f(n) \in \mathcal{F}_A$ for some constant $A \geq 1$. Applying Theorem 1, we deduce that

$$\sum_{\substack{n \leq N \\ n \in \mathcal{B}_{\alpha, \beta}}} \frac{r_4(n)}{n} = \alpha^{-1} \sum_{n \leq N} \frac{r_4(n)}{n} + O\left(\frac{N \log \log N}{\log N}\right),$$

where the implied constant depends only on α .

From the asymptotic formula (see for example [4, p22]):

$$\sum_{n \leq N} r_4(n) = \frac{\pi^2 N^2}{2} + O(N \log N),$$

we have by partial summation:

$$\sum_{n \leq N} \frac{r_4(n)}{n} = \pi^2 N + O((\log N)^2).$$

Consequently,

$$\sum_{\substack{n \leq N \\ n \in \mathcal{B}_{\alpha, \beta}}} \frac{r_4(n)}{n} = \alpha^{-1} \pi^2 N + O\left(\frac{N \log \log N}{\log N}\right).$$

Using partial summation once more, we obtain the statement of Corollary 3.

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